J. DIFFERENTIAL GEOMETRY
45 (1997) 547-574

MANIN TRIPLES FOR LIE BIALGEBROIDS

ZHANG-JU LIU, ALAN WEINSTEIN & PING XU

Abstract

In his study of Dirac structures, a notion which includes both Poisson structures and closed 2-forms, T. Courant introduced a bracket on the direct sum of vector fields and 1-forms. This bracket does not satisfy the Jacobi identity except on certain subspaces. In this paper we systematize the properties of this bracket in the definition of a **Courant algebroid**. This structure on a vector bundle $E \to M$, consists of an antisymmetric bracket on the sections of E whose "Jacobi anomaly" has an explicit expression in terms of a bundle map $E \to TM$ and a field of symmetric bilinear forms on E. When M is a point, the definition reduces to that of a Lie algebra carrying an invariant nondegenerate symmetric bilinear form.

For any Lie bialgebroid (A, A^*) over M (a notion defined by Mackenzie and Xu), there is a natural Courant algebroid structure on $A \oplus A^*$ which is the Drinfel'd double of a Lie bialgebra when M is a point. Conversely, if A and A^* are complementary isotropic subbundles of a Courant algebroid E, closed under the bracket (such a bundle, with dimension half that of E, is called a **Dirac structure**), there is a natural Lie bialgebroid structure on (A, A^*) whose double is isomorphic to E. The theory of Manin triples is thereby extended from Lie algebras to Lie algebroids.

Our work gives a new approach to bihamiltonian structures and a new way of combining two Poisson structures to obtain a third one. We also take some tentative steps toward generalizing Drinfel'd's theory of Poisson homogeneous spaces from groups to groupoids.

1. Introduction

The aim of this paper is to solve, in a unified way, several mysteries which have arisen over the past few years in connection with generalizations of the notion of Lie algebra in differential geometry.

Received July 18, 1996. The first author was partially supported by NSF of China and Math. Center of SEC, the second author by NSF grant DMS93-09653, and the third author by NSF grant DMS95-04913 and an NSF postdoctoral fellowship.