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# THE CLASS OF THE DIAGONAL IN FLAG BUNDLES

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## 1. Introduction

Let $G$ be a complex reductive group and $B$ a Borel subgroup of $G$, and let $BG$ and $BB$ denote the classifying spaces of these groups. Then $BB \to BG$ is a flag bundle: a fibration with fibers isomorphic to the flag variety $G/B$. The diagonal $\Delta \subset BB \times_{BG} BB$ can be used to define a class in $H^*(BB \times_{BG} BB)$. For classical groups this class has been studied by Fulton and by Pragacz and Ratajski (see [14], [15], [21], [22]). This paper studies the class of the diagonal for general $G$, from a Lie-theoretic point of view. Here $H^*$ denotes cohomology with complex coefficients; see Section 2 for a discussion of integer cohomology and Chow groups.

The motivation for this study comes from degeneracy loci. In its simplest form, if $V$ and $W$ are vector bundles on $M$ and $\phi : V \to W$ a generic bundle map, the locus $Z \subset M$ where $\phi_z$ has less than maximal rank is called a degeneracy locus. More generally one can consider a vector bundle $V$ equipped with a pair of flags of subbundles $E_1 \subset \ldots \subset E_n = V$ and $F_1 \subset \ldots \subset F_n = V$. This corresponds to $G = GL_n$. For each $w \in S_n$ (the Weyl group of $GL_n$) there is a locus $Z_w \subset M$ defined by certain incidence relations between the flags $E$ and $F$. Bundles equipped with an orthogonal or symplectic form, and isotropic or Lagrangian flags of subbundles, correspond to the groups $SO(n)$ or $Sp(n)$. Many schemes can be realized as degeneracy loci and for this reason general facts about such loci are useful (see [23] for a survey, and also [9], [16]).

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