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PRIMITIVE CALABI-YAU THREEFOLDS

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0. Introduction

A Calabi-Yau threefold is a complex projective threefold X (possibly with some suitable class of singularities, say terminal or canonical) with $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. One of the fundamental gaps in the classification of algebraic threefolds is the lack of understanding of Calabi-Yau threefolds. Here I will try to set forth a program to bring the morass of thousands of examples of Calabi-Yaus under control.

The ideas here go back to the papers of Friedman [4] and Reid [27]. Friedman studied smoothability of Calabi-Yau threefolds with ordinary double points. Based on these results, Reid conjectured that there could perhaps be a single irreducible moduli space of (non-Kähler) Calabi-Yau threefolds, such that any Calabi-Yau threefold is the small resolution of a degeneration of this family to something with ordinary double points. So one can think of all the chaos of the algebraic examples as simply being "boundary phenomena" for the moduli space of this master Calabi-Yau. I suspect the most difficult part of this conjecture, often known as Reid's fantasy, will be passing from algebraic to non-algebraic threefolds. We do not understand how to deal with non-Kähler Calabi-Yau threefolds or find non-algebraic contractions.

Unlike in the K3 case, where it is possible to deform an algebraic K3 surface to a non-algebraic one, the deformation of a projective Calabi-Yau threefold, even singular, is still projective. So it makes sense to insist on staying within the projective category. Reid's picture given

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