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RIEMANN-ROCH FOR TORIC ORBIFOLDS

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1. Introduction

Let $\alpha_1, \ldots, \alpha_d$ and μ be elements of the integer lattice, $\mathbf{Z}^n$, and let $N(\mu)$ be the number of solutions, $k = (k_1, \ldots, k_d)$, of the equation

$$k_1\alpha_1 + \ldots + k_d\alpha_d = \mu \ , \tag{1.1}$$

the k_i's being non-negative integers. For this equation to be well-posed we will assume that the α_i's lie in a fixed open half-space. In other words: for all i, $\xi(\alpha_i) > 0$, for some $\xi \in (\mathbf{R}^n)^*$. (Otherwise, for every μ for which (1.1) admits a solution it will admit an infinite number of solutions!) Also, in order for (1.1) to be solvable, μ has to be contained in the lattice generated by the α_i's, and, with no essential loss of generality, we can assume that this lattice is $\mathbf{Z}^n$ itself.

For every subset, I, of $\{1, \ldots, d\}$ let $\mathbf{R}^I$ be the subspace of $\mathbf{R}^n$ spanned by those α_i's for which i is in I. We will say that μ is in *general position* with respect to $\alpha_1, \ldots, \alpha_d$ if $\mu \in \mathbf{R}^I \leftrightarrow \mathbf{R}^I = \mathbf{R}^n$. (Thus the elements of $\mathbf{R}^I$ are *not* in general position with respect to α_1, $\ldots$, α_d if $\mathbf{R}^I$ is a proper subspace of $\mathbf{R}^n$.)

Let us consider the real analogue of (1.1):

$$s_1\alpha_1 + \ldots + s_d\alpha_d = \mu + \epsilon \quad , \quad \epsilon \in \mathbf{R}^n, \tag{1.2}$$

the s_i's being non-negative *real* numbers. The set of solutions, s, of this equation is a convex polytope in $\mathbf{R}^d$. We will denote this polytope by $\Delta_{\mu+\epsilon}$ and its I-th face:

$$\Delta^I_{\mu+\epsilon} = \{s = (s_1, \ldots, s_d) \in \Delta_{\mu+\epsilon},\ s_i = 0 \text{ for } i \in I\} \tag{1.3}$$

by $\Delta^I_{\mu+\epsilon}$. We claim:

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