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WHITNEY FORMULA IN HIGHER DIMENSIONS

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Abstract

The classical Whitney formula relates the algebraic number of times that a generic immersed plane curve cuts itself to the index ("rotation number") of this curve. Both of these invariants are generalized to higher dimension for the immersions of an n-dimensional manifold into an open $(n+1)$-manifold with the null-homologous image. We give a version of the Whitney formula if n is even. We pay special attention to immersions of S^2 into $\mathbb{R}^3$. In this case the formula is stated in the same terms which were used by Whitney for immersions of S^1 into $\mathbb{R}^2$.

1. Introduction

Let $f : S^1 \to \mathbb{R}^2$ be a generic immersion (i.e., an immersion without triple points and self-tangencies). The *index* of f is the degree of the Gauss map (which maps S^1 to the direction of $df(v)$ where v is a tangent vector field positive with respect to the standard orientation of S^1). Whitney in [7] showed that the index is the only invariant of f up to deformation in the class of immersions.

Fix a generic point $x \in S^1$. The cyclic order on S^1 determined by the orientation defines a linear order on $S^1 - \{x\}$. This determines an ordering of the positive vectors tangent to the two branches of f at every double point d of f. Following Whitney we define the sign $\epsilon_x(d)$ of d to be $+1$ (resp. -1) if the frame composed of these tangent vectors is *negative* (resp. *positive*) in $\mathbb{R}^2$.

We define the function ind : $\mathbb{R}^2 \to \frac{1}{2}\mathbb{Z}$ in the following way. The (integer) value of ind at $y \in \mathbb{R}^2 - f(S^1)$ is defined as the linking number of the oriented cycle $f(S^1)$ and the 0-dimensional cycle composed of the point y taken with the positive orientation and a point near infinity taken with the negative

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