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SHORT TIME BEHAVIOR OF THE HEAT KERNEL AND ITS LOGARITHMIC DERIVATIVES

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Abstract

Let M be a compact, connected Riemannian manifold, and let $p_t(x,y)$ denote the fundamental solution to Cauchy initial value problem for the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$, where Δ is the Levi–Civita Laplacian. The purpose of this note is to study the asymptotic behavior of derivatives of $\log p_t(\,\cdot\,,y)$ at x as $t \searrow 0$. In particular, we show that a dramatic change takes place when x moves inside the cut-locus of y.

0. Introduction

Let M be a compact, connected, d-dimensional Riemannian manifold, denote by $\mathcal{O}(\mathcal{M})$ with fiber map $\pi : \mathcal{O}(\mathcal{M}) \longrightarrow M$ the associated bundle of orthonormal frames $\mathfrak{e}$, and use the Levi–Civita connection to determine the horizontal subspace $H_{\mathfrak{e}}\big(\mathcal{O}(\mathcal{M})\big)$ at each $\mathfrak{f} \in \mathcal{O}(\mathcal{M})$. Next, given $\mathbf{v} \in \mathbb{R}^d$, let $\mathfrak{E}(\mathbf{v})$ be the *basic vector field* on $\mathcal{O}(\mathcal{M})$ determined by properties that

$$\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} \in H_{\mathfrak{e}}\big(\mathcal{O}(\mathcal{M})\big) \quad \text{and} \quad d\pi\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} = \mathfrak{e}\mathbf{v} \quad \text{for all } \mathfrak{e} \in \mathcal{O}(\mathcal{M}).$$

(Here, and whenever convenient, we think of $\mathfrak{e}$ as an isometry from $\mathbb{R}^d$ onto $T_{\pi(\mathfrak{e})}(M)$.) In particular, if $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the standard orthonormal basis in $\mathbb{R}^d$, then we set $\mathfrak{E}_k(\mathfrak{e}) = \mathfrak{E}(\mathbf{e}_k)_{\mathfrak{e}}$. If, for $\mathcal{O} \in O(d)$ (the orthogonal group on $\mathbb{R}^d$) $R_{\mathcal{O}} : \mathcal{O}(\mathcal{M}) \longrightarrow \mathcal{O}(\mathcal{M})$ is defined so that

$$R_{\mathcal{O}}\mathfrak{e}\,\mathbf{v} = \mathfrak{e}\,\mathcal{O}\mathbf{v}, \quad \mathfrak{e} \in \mathcal{O}(\mathcal{M}) \text{ and } \mathbf{v} \in \mathbb{R}^d,$$

then it easy to check that

$$(0.1) \qquad dR_{\mathcal{O}}\mathfrak{E}(\mathbf{v})_{\mathfrak{e}} = \mathfrak{E}\big(\mathcal{O}^{\top}\mathbf{v}\big)_{R_{\mathcal{O}}\mathfrak{e}}, \quad \mathfrak{e} \in \mathcal{O}(\mathcal{M}) \text{ and } \mathbf{v} \in \mathbb{R}^d.$$

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