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THE SYMPLECTIC GEOMETRY OF POLYGONS IN EUCLIDEAN SPACE

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Abstract

We study the symplectic geometry of moduli spaces M_r of polygons with fixed side lengths in Euclidean space. We show that M_r has a natural structure of a complex analytic space and is complex-analytically isomorphic to the weighted quotient of $(S^2)^n$ by $PLS(2,\mathbb{C})$ constructed by Deligne and Mostow. We study the Hamiltonian flows on M_r obtained by bending the polygon along diagonals and show the group generated by such flows acts transitively on M_r. We also relate these flows to the twist flows of Goldman and Jeffrey-Weitsman.

1. Introduction

Let $\mathcal{P}_n$ be the space of all n-gons with distinguished vertices in Euclidean space $\mathbb{E}^3$. An n-gon P is determined by its vertices $v_1, ..., v_n$. These vertices are joined in cyclic order by edges $e_1, ..., e_n$ where e_i is the oriented line segment from v_i to v_{i+1}. Two polygons $P = (v_1, ..., v_n)$ and $Q = (w_1, ..., w_n)$ are identified if and only if there exists an orientation preserving isometry g of $\mathbb{E}^3$ which sends the vertices of P to the vertices of Q, that is

$$gv_i = w_i, \ 1 \leq i \leq n$$

Let $r = (r_1, ..., r_n)$ be an n-tuple of positive real numbers. Then M_r is defined to be the space of n-gons with side lengths $r_1, ..., r_n$ modulo isometries as above. The group $\mathbb{R}_+$ acts on $\mathcal{P}_n$ by scaling and we obtain an induced isomorphism $M_r \cong M_{\lambda r}$ for $\lambda \in \mathbb{R}_+$. Thus we lose nothing by assuming

$$\sum_{i=1}^{n} r_i = 2$$

We make this normalization to agree with [4], §2.

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