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GLOBAL ASYMPTOTIC LIMIT OF SOLUTIONS OF THE CAHN–HILLIARD EQUATION

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Abstract

We study the asymptotic limit, as $\varepsilon \searrow 0$, of solutions of the Cahn–Hilliard equation

$$u_t^\varepsilon = \Delta(-\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon))$$

under the assumption that the initial energy

$$\int_\Omega \left[\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,0)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,0))\right]$$

is bounded independent of ε. Here $f = F'$, and F is a smooth function taking its global minimum 0 only at $u = \pm 1$. We show that there is a subsequence of $\{u^\varepsilon\}_{0<\varepsilon\le 1}$ converging to a weak solution of an appropriately defined limit Cahn–Hilliard problem. We also show that, in the case of radial symmetry, all the interfaces of the limit have multiplicity one for almost all time $t > 0$, regardless of initial energy distributions.

1. Introduction

In this paper, we shall study the asymptotic limit, as $\varepsilon \searrow 0$, of the solutions of the Cahn–Hilliard equation

$$\text{(1.1)} \quad \begin{cases} u_t^\varepsilon(x,t) = \Delta v^\varepsilon(x,t), & (x,t) \in \Omega \times (0,\infty), \\ v^\varepsilon = -\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon), & (x,t) \in \Omega \times [0,\infty), \\ \frac{\partial}{\partial n}u^\varepsilon = \frac{\partial}{\partial n}v^\varepsilon = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), & x \in \Omega. \end{cases}$$

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