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KILLING FIELDS IN COMPACT LORENTZ 3-MANIFOLDS

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Abstract

Here we classify flows on compact 3-manifolds that preserve smooth Lorentz metrics.

1. Introduction

The geodesic and horocyclic flows on the unit tangent bundle of a hyperbolic surface are well known by their beautiful, but very different properties. Nervertheless, these two flows with antagonistic dynamics are unified by the Lorentz geometry. By this, we mean that both of them are Killing fields for Lorentz structures. The purpose of this paper is to show that Lorentz geometry not only unifies but also characterize them. That is, the nontrivial (i.e., nonequicontinous) Killing fields for Lorentz metrics in dimension three, are all "derived from" geodesic or horocyclic flows.

Algebraically, the unit tangent bundle of the 2–hyperbolic space is identified with the group $PSL(2,\mathbf{R})$. The fundamental group of a hyperbolic surface is thus identified with a discrete subgroup Γ in $PSL(2,\mathbf{R})$, and its unit tangent bundle with $\Gamma\backslash\ PSL(2,\mathbf{R})$. A one-parameter group $\{f^t\}$ in $PSL(2,\mathbf{R})$ determines on $\Gamma\backslash\ PSL(2,\mathbf{R})$ a right translation flow $\Gamma x \to \Gamma x f^t$. The geodesic (resp. horocyclic) flow corresponds to the hyperbolic (resp. parabolic) one-parameter group: $g^t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}$ (resp. $h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$) . In fact any noncompact one-parameter of $PSL(2,\mathbf{R})$ is conjugate to $\{g^{\alpha t}\}$ or $\{h^{\alpha t}\}$ for some real α. If a one-parameter group is compact, it is conjugate to $\left\{\begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix}\right\}$.

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