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LEVEL SET APPROACH TO MEAN CURVATURE FLOW IN ARBITRARY CODIMENSION

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Abstract

We develop a level set theory for the mean curvature evolution of surfaces with arbitrary co-dimension, thus generalizing the previous work [8, 15] on hypersurfaces. The main idea is to surround the evolving surface of codimension-k in $\mathbf{R}^d$ by a family of hypersurfaces (the level sets of a function) evolving with normal velocity equal to the sum of the $(d-k)$ smallest principal curvatures. The existence and the uniqueness of a weak (level-set) solution is easily established by using mainly the results of [8] and the theory of viscosity solutions for second order nonlinear parabolic equations. The level set solutions coincide with the classical solutions whenever the latter exist. The proof of this connection uses a careful analysis of the squared distance from the surfaces. It is also shown that varifold solutions constructed by Brakke [7] are included in the level-set solutions. The idea of surrounding the evolving surface by a family of hypersurfaces with a certain property is related to the barriers of De Giorgi. An introduction to the theory of barriers and its connection to the level set solutions is also provided.

1. Introduction

Recently, Evans & Spruck [15] and, independently, Chen, Giga & Goto [8] developed a level set approach for hypersurfaces evolving by their mean curvature. We extend this approach to surfaces with arbitrary co-dimension.

In the classical setup, mean curvature flow is a geometric initial value problem. Starting from a smooth initial surface Γ_0 in $\mathbf{R}^d$, the solution Γ_t evolves in time so that at each point its normal velocity vector is equal to its mean curvature vector. By parametric methods of differential geometry much has been obtained for convex or graph-like initial surfaces or for planar curves. See for instance Altschuler & Grayson [3], Ecker & Huisken [13], Gage & Hamilton [21], Grayson [23], and Huisken [25]. However for $d \geq 3$, initially smooth surfaces may develop geometric singularities. For example the dumbbell region in $\mathbf{R}^3$ splits into two

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