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RATIONALITY OF SECONDARY CLASSES

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Abstract

We prove the Bloch conjecture : $c_2(E) \in H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ is torsion for holomorphic rank-two vector bundles E with an integrable connection over a complex projective variety X. We prove also the rationality of the Chern-Simons invariant of compact arithmetic hyperbolic three-manifolds. We give a sharp higher-dimensional Milnor inequality for the volume regulator of all representations to $PSO(1,n)$ of fundamental groups of compact n-dimensional hyperbolic manifolds, announced in our earlier paper.

1. The theorem

1.1. Let X be a smooth complex projective variety. Consider a representation $\rho : \pi_1(X) \to SL(2, \mathbb{C})$. Let E_ρ be the corresponding rank-two vector bundle over X. Viewing E_ρ as an algebraic vector bundle, denote by $c_2(E_\rho)$ the second Chern class in Deligne cohomology group $H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$ ([15], [20]). Recall that there is an exact sequence $0 \to J^2(X) \to H^4_{\mathcal{D}}(X, \mathbb{Z}(2)) \to H^4(X, \mathbb{Z}(2))$, and by the Chern–Weil theory, the image of $c_2(E_\rho)$ in $H^4(X, \mathbb{Z}(2))$ is torsion. Therefore $c_2(E_\rho)$ lies in the image of $H^3(X, \mathbb{C}/\mathbb{Z})$ under the natural map $H^3(X, \mathbb{C}/\mathbb{Z}) \to H^3(X, \mathbb{C}/\mathbb{Z}(2)) \to H^4_{\mathcal{D}}(X, \mathbb{Z}(2))$. It was proved by Bloch [3], Gillet–Soulé [24] and Soulé [50] that in fact, $c_2(E_\rho)$ is an image of the secondary characteristic class $Ch(\rho)$ of a flat bundle E_ρ (equivalently, of a representation ρ), lying in $H^3(X, \mathbb{C}/\mathbb{Z})$.The $\mathbb{R}/\mathbb{Z}$-part of this class was introduced and studied by Chern–Simons [9] and Cheeger–Simons [8],and will be called Cheeger–Chern–Simons class and denoted $ChS(\rho)$.The $\mathbb{R}$-part lying in $H^3(X, \mathbb{R})$ will be called Borel hyperbolic volume class (regulator) and denoted $Vol(\rho)$. Remark that if ρ is unitary, then $Vol(\rho) = 0$. Next, for a field F denote $\mathcal{B}(F)$ the Bloch group of F. Recall that for F algebraically closed there is an exact sequence $0 \to \mu_F^{\otimes 2} \to H_3(SL(2, F), \mathbb{Z}) \to \mathcal{B}(F) \to 0$ of Bloch–Wigner–Dupont–Sah [19]. The dilogarithm function of Bloch–Wigner defines a homomorphism $D : \mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q} = \mathbb{R}/\mathbb{Q} \oplus i\mathbb{R}$ which splits to

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