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THE LENGTH OF A CUT LOCUS ON A SURFACE AND AMBROSE'S PROBLEM

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1. Introduction

There are many results about the cut locus $C(p)$ of a point p on a surface (M, g) going back to H.Poincaré's old paper [8]. S.Myers proved that if M is a real analytic sphere, $C(p)$ is a finite tree each of whose edges is an analytic curve with finite length [9]. It follows that the total length (1-dimensional Hausdorff measure) of $C(p)$ is finite. In the case of a C^∞ surface, $C(p)$ is somewhat complicated. In [3] H.Gluck and D.Singer constructed a C^∞ metric on S^2 so that there is a point p whose cut locus has infinitely many edges sharing a common end point and thus is not triangulable. Even in this case the total length of $C(p)$ is finite. Recently K.Shiohama and M.Tanaka showed that even on an Alexsandrov surface the cut locus of a point carries the structure of a local tree [10]. It is easy to construct an Alexandrov sphere so that the total length of a cut locus is infinite.

The purpose of this article is to study the relation between the length of a cut locus of a surface and the regularity of its metric. In the following, we will answer the question "When does $C(p)$ have infinite total length (1-dimensional Hausdorff measure) ?".

Theorem A. *Suppose (M, g) is a complete surface with a Riemannian metric of class C^2. Then any compact subset of the cut locus of $p \in M$ has finite 1-dimensional Hausdorff measure.*

Theorem B. *There is a $C^{1,1}$ metric on S^2 so that there is a point $p \in S^2$ whose cut locus $C(p)$ has infinite total length (1-dimensional Haudorff measure).*

In particular in the case of a compact surface, if the metric has C^2 regularity, the total lengths of the cut loci are all finite. If the metric loses C^2 regularity, then the cut loci may have infinite total length, and can further become what we know as a fractal set [7]. In the proof of Theorem A, we will show that the function, which assigns to each initial

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