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VOLUME INCREASING ISOMETRIC DEFORMATIONS OF CONVEX POLYHEDRA

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Abstract

In this paper, we prove that the surface of a generic convex polyhedron in E^3 can be isometrically deformed so as to enclose greater volume. Our deformed surfaces have at least seven times more faces than the original surface. They are extrinsically nonconvex, but have the same intrinsic geometry (e.g., area, angle deficits, etc.). Roughly put, the deformations are obtained by simultaneously delivering karate chops to the edges of the polyhedron. While the edges implode, to preserve the intrinsic geometry, portions of the faces move outward, leading to a net increase of volume. The regular tetrahedron can be isometrically deformed to enclose over 37.7% more volume, while the cube, octahedron, dodecahedron, and icosahedron enjoy increases over 21.8%, 11.5%, 9.3%, and 3.6%, respectively. Many questions have arisen. In particular, what is the nature of an isometric embedding, if any, of an abstract surface which encloses the maximum volume? We propose that the term "sandbag" be used to describe such an embedding. What kinds of singularities can they have? Are they unique up to Euclidean motions?

1. Introduction

A *deformation* of a surface S in Euclidean 3–space E^3 is a continuous map $h : S \times [0, \varepsilon] \longrightarrow E^3$, such that for $h_t(\cdot) := h(\cdot\,, t)$, we have that h_0 is the inclusion of S in E^3. The deformation h is an *isometric deformation* if the length of any rectifiable curve γ in S is constant under the deformation; i.e., $L(h_t \circ \gamma) = L(\gamma)$, for all $t \in [0, \varepsilon]$. Thus, the intrinsic geometry (in particular, the area and Gaussian curvature measure of S) is preserved under an isometric deformation. In our main results, the surface S will be a generic convex polyhedral surface with triangular faces, such that each pair of faces with a common edge meet at a dihedral angle strictly less than π. However, we also consider the cube and dodecahedron. It seems likely that the assumption of triangular faces is only a technicality and probably can be dropped. For the specific isometric deformation h that we will construct, for $t > 0$, $h_t(S)$ will also be a polyhedral surface, but with 7 times as many true faces

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