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CONTRACTION OF CONVEX HYPERSURFACES BY THEIR AFFINE NORMAL

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Abstract

An affine-invariant evolution equation for convex hypersurfaces in Euclidean space is defined by assigning to each point a velocity equal to the affine normal vector. For an arbitrary compact, smooth, strictly convex initial hypersurface, it is shown that this deformation produces a unique, smooth family of convex hypersurfaces, which converge to a point in finite time. Furthermore, the hypersurfaces converge smoothly to an ellipsoid after rescaling about the final point to make the enclosed volume constant. The result leads to simple proofs of some affine-geometric isoperimetric inequalities.

1. Introduction

Consider a smooth, strictly convex hypersurface in Euclidean space $\mathbf{R}^{n+1}$ given by a smooth embedding $\varphi_0 : S^n \to \mathbf{R}^{n+1}$, where S^n is the unit sphere in $\mathbf{R}^{n+1}$. We consider the evolution of such an embedding to produce a family of embeddings $\varphi : S^n \times [0,T) \to \mathbf{R}^{n+1}$ satisfying the following equation:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(z,t) &= \mathcal{N}(z,t) \\ \varphi(z,0) &= \varphi_0(z) \end{aligned} \tag{1.1}$$

for all z in S^n and t in $[0,T)$. Here $\mathcal{N}$ is the affine normal vector. This equation is the unique second order parabolic evolution equation for hypersurfaces in Euclidean space which is invariant under affine transformations of $\mathbf{R}^{n+1}$: Suppose $\varphi : S^n \otimes [0,T) \to \mathbf{R}^{n+1}$ is a solution of Equation (1.1), and $L : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is an affine transformation with the modulus of its determinant $|L| > 0$. Then the family of embeddings $\varphi_L : S^n \otimes [0, |L|^{\frac{2}{n+2}}T) \to \mathbf{R}^{n+1}$ given by $\varphi_L(z,t) = L \circ \varphi(z, |L|^{-\frac{2}{n+2}}t)$ is a solution to Equation (1.1) with initial condition $(\varphi_L)_0 = L \circ \varphi_0$.

Note that the round sphere in $\mathbf{R}^{n+1}$ is a homothetic solution of Equation (1.1): If we have $\varphi_0(z) = r_0 z$ as initial condition, then the solution

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