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DONALDSON INVARIANTS OF 4-MANIFOLDS WITH SIMPLE TYPE

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1. Introduction

The Donaldson invariant of a smooth simply connected 4-manifold X with odd $b^+ \geq 3$ is a linear map

$$D_X : \mathbf{A}(X) = \mathrm{Sym}_*(H_0(X) \oplus H_2(X)) \to \mathbf{R}$$

defined on the graded algebra $\mathbf{A}(X)$, where elements of $H_i(X)$ are defined to have degree $\frac{1}{2}(4-i)$. Since its presentation by Simon Donaldson [8], this invariant has proven indispensible for distinguishing smooth 4-manifolds with the same homotopy type. Roughly, if $x \in H_0(X)$, $\alpha \in H_2(X)$, and $z = \alpha^a x^b \in \mathbf{A}(X)$ has degree d, one can define D_X by the formula

$$D_X(z) = \langle \mu(\alpha)^a \mu(x)^b, [\mathcal{M}_X^{2d}] \rangle,$$

where $[\mathcal{M}_X^{2d}]$ is the fundamental class of the (compactified) $2d$-dimensional moduli space of anti-self-dual connections on an $SU(2)$ bundle over X, and $\mu : H_*(X) \to H^{4-*}(\mathcal{M}_X^{2d})$ is a canonical homomorphism. The instanton moduli spaces $\mathcal{M}_X^{2d}$ have formal dimensions congruent to $-3(1+b_X^+)$ (mod 4), and D_X is defined to be 0 in degrees other than $\frac{1}{2}(1+b_X^+)$ (mod 4).

Despite its utility, the Donaldson invariant has proven difficult to evaluate, and its general form has remained elusive. In this paper we investigate the general structure of this invariant through a study of its behavior in the presence of embedded spheres. A turning point in the study of the invariants arose with the results of P. Kronheimer and T. Mrowka [24] concerning the structure of the Donaldson invariants under the technical assumption of "simple type." This assumption states essentially that for the generator x of $H_0(X)$ and arbitrary $z \in \mathbf{A}(X)$, $D_X(x^2 z) = 4 D_X(z)$. Their results are obtained through a study of

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