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LOCAL POSITIVITY OF AMPLE LINE BUNDLES

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Introduction

The purpose of this paper is to establish a lower bound on the Seshadri constants measuring the local positivity of an ample line bundle at a general point of a complex projective variety of arbitrary dimension.

Let X be an irreducible complex projective variety, and let L be a nef line bundle on X. Demailly [6] has introduced a very interesting invariant which in effect measures how positive L is locally near a given smooth point $x \in X$. This *Seshadri constant* $\epsilon(L, x) \in \mathbf{R}$ may be defined as follows. Consider the blowing up

$$f : Y = \mathrm{Bl}_x(X) \longrightarrow X$$

of X at x, and denote by $E = f^{-1}(x) \subset Y$ the exceptional divisor. Then f^*L is a nef line bundle on Y, and we put

$$\epsilon(L, x) = \sup \{ \epsilon \geq 0 \mid f^*L - \epsilon \cdot E \text{ is nef} \}.$$

Here $f^*L - \epsilon E$ is considered as an $\mathbf{R}$-divisor on Y, and to say that it is nef means simply that $f^*L \cdot C' \geq \epsilon E \cdot C'$ for every irreducible curve $C' \subset Y$. For example, if L is very ample, then $\epsilon(L, x) \geq 1$ for every smooth point $x \in X$. Seshadri's criterion (cf. [10 (Chapter 1)]) states that L is ample if and only if there is a positive number $\epsilon > 0$ such that $\epsilon(L, x) \geq \epsilon$ for every $x \in X$. We refer to Section 1 below, as well as [6 (§6)], for alternative characterizations and further properties of Seshadri constants.

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