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TWISTOR SPACES, EINSTEIN METRICS AND ISOMONODROMIC DEFORMATIONS

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1. Introduction

The characteristic feature of twistor theory is its ability to convert questions in differential geometry and differential equations into equivalent ones in algebraic geometry. Moreover, the natural objects in algebraic geometry such as sheaf cohomology groups or vector bundles correspond in a remarkably fortuitous way to solutions of equations which have at some level a physical or geometrical significance. The fundamental example of this is the basic correspondence between an anti-self-dual conformal structure on a 4-manifold M and the holomorphic structure of a complex 3-manifold Z, its twistor space.

In this paper we shall study in depth a problem which goes one step further: to describe an anti-self-dual conformal structure not by algebraic geometry, but by topology, or indeed algebra – the representations of a fundamental group. The conformal structures which are amenable to this approach are those which admit $SU(2)$ as a symmetry group, with certain generic properties. Each such structure describes, and is determined by, a representation in $SL(2,\mathbf{C})$ of a free group on 3 generators. Building on the basic framework of this correspondence, we can introduce other geometrical structures, and in particular an Einstein metric in the conformal class. It turns out that the Einstein condition yields a considerable simplification of the representation, so much so that we can retrace our footsteps from the topology back to the differential geometry and write down new explicit solutions of anti-self-dual $SU(2)$-invariant Einstein metrics with 3-dimensional generic orbits. Among these are two families of complete metrics on the unit ball in $\mathbf{R}^4$: one two-parameter family consists of deformations of the

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