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ON THE GLUING PROBLEM FOR THE η-INVARIANT

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Abstract

We solve the gluing problem for the η-invariant. Consider a generalized Dirac operator D over a compact Riemannian manifold M that is partitioned by a compact hypersurface N such that $M := M_1 \cup_N M_2$. We assume that the Riemannian metric of M and D have a product structure near N, i.e., $D = I(\partial/\partial\tau + D_N)$ with some Dirac operator D_N on N. Using boundary conditions of Atiyah-Patodi-Singer type parametrized by Lagrangian subspaces L_i of $\ker D_N$ we define selfadjoint extensions D_i, $i = 1, 2$, over M_i. We express the η-invariant of D in terms of the η-invariants of D_i, an invariant $m(L_1, L_2)$ of the pair of the Lagrangian subspaces L_1, L_2, which is related to the Maslov index and an integer-valued term J. In the adiabatic limit, i.e., if a tubular neighborhood of N is long enough, the vanishing of J is shown under certain regularity conditions. We apply this result in order to prove cutting and pasting formulas for the η-invariant, a Wall non-additivity result for the index of Atiyah-Patodi-Singer boundary value problems and a splitting formula for the spectral flow.

1. The gluing problem and applications

1.1. Introduction. We solve the gluing problem for the η-invariant of generalized Dirac operators. Consider a closed, compact Riemannian manifold carrying a Dirac bundle with a generalized Dirac operator. Assume that this manifold is separated into two pieces by a compact hypersurface. The gluing problem for the η-invariant consists in expressing the η-invariant of the original Dirac operator in terms of the η-invariants of the Dirac operators living on the pieces.

The selfadjoint operators on the two components with boundary depend on a boundary condition given by Lagrangian subspaces L_1, L_2 of a certain symplectic vector space. The gluing formula also contains an additional real-valued term $m(L_1, L_2)$, which is nicely related to the symplectic geometry and the Maslov index.

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