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A NOTE ON THE KAZDAN-WARNER TYPE CONDITIONS

WENXIONG CHEN & CONGMING LI

Abstract

We consider prescribing Gaussian curvature on a 2-sphere S^2. There are well-known Kazdan-Warner and Bourguinon-Ezin necessary conditions for a function K to be the Gaussian curvature of some pointwise conformal metric. Then are those necessary conditions also sufficient? This is a problem of common concern and has been left open for a few years. In this paper, we answer the question negatively. First, we show that if K is rotationally symmetric and is monotone in the region where $K > 0$, then the problem has no rationally symmetric solution. Then we provide a family of functions K satisfying the Kazdan-Warner and Bourguinon-Ezin conditions, for which the problem has no solution at all. We also consider prescribing scalar curvature on S^n for $n \geq 3$. We prove the nonexistence of rationally symmetric solution for the above-mentioned functions.

1. Introduction

Given a continuous function $K(x)$ on a compact surface S, it is interesting to know that whether it can be the Gaussian curvature of some metric. In practice, one often seeks the unknown metric by picking a basic metric g_0, and then pointwise conformally deforms it to the desired metric g. If we let $g = e^{2u} g_0$, then it is equivalent to solving the following nonlinear elliptic equation:

$$-\Delta_0 u + K_0(x) = K(x) e^{2u(x)}, \qquad x \in S, \tag{1.1}$$

where Δ_0 and $K_0(x)$ are the Laplacian and the Gaussian curvature of g_0. In the last few years, a lot of work has been done to understand this (cf. [2], $\cdots$, [8], [11], $\cdots$, [19] and the references therein).

For equation (1.1) to have a solution, the function $K(x)$ must satisfy the obvious Gauss-Bonnet sign condition

$$\int_S K(x) e^{2u} \, dx = 2\pi\chi, \tag{1.2}$$

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