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HARNACK ESTIMATE FOR THE MEAN CURVATURE FLOW

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1. The result

We consider the evolution of a hypersurface M^n in Euclidean space R^{n+1} by its mean curvature. This was first studied by Huisken [3]. In this flow each point Y on M moves in the direction of the unit normal vector N with velocity equal to the mean curvature H, the trace of the second fundamental form $H(V, V)$ over the tangent vectors V. We confine our attention to solutions which are smooth and either compact, or else are complete with bounded second fundamental form.

1.1. Main Theorem A. *For any weakly convex solution to the mean curvature flow for $t > 0$ we have*

$$\frac{\partial H}{\partial t} + \frac{1}{2t}H + 2DH(V) + H(V, V) \geq 0$$

for all tangent vectors V.

This is the differential Harnack inequality for the mean curvature flow. As usual (see Li and Yau [4]) we can integrate over paths in space-time to get an integral Harnack inequality.

1.2. Corollary. *For any weakly convex solution to the mean curvature flow for $t > 0$ we have*

$$H(Y_2, t_2) \geq \sqrt{t_1/t_2}\, e^{-\Delta/4} H(Y_1, t_1)$$

for any two points Y_1 and Y_2 on the evolving surface at times t_1 and t_2 with $0 < t_1 < t_2$, where

$$\Delta = \inf \int \left| \frac{dY}{dt} \right|_M^2 dt$$

is the infimum over all paths $Y(t)$ remaining on the surface at time t with $Y = Y_1$ at $t = t_1$ and $Y = Y_2$ at $t = t_2$, dY/dt is the velocity vector

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