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ON AFFINE CRYSTALLOGRAPHIC GROUPS

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Dedicated to Professor J. L. Mennicke, on the occasion of his 60th birthday

1. Introduction

Setting the scene. An *affine crystallographic group* (ACG) is a properly discontinuous group Γ of affine transformations on some (finite-dimensional) real vector space V, such that the quotient space $\Gamma\backslash V$ is compact. If Γ is also torsion free, then $\Gamma\backslash V$ is a compact *affine space form*, with fundamental group isomorphic to Γ; every flat, complete, compact connected differentiable manifold arises this way [24, Corollary 1.9.6]. If $\Gamma \leq A$, where A is a given subgroup of $\mathrm{Aff}(V)$, the group of all affine transformations of V, we shall call Γ an ACG *of type* A (for example, when A is the group of all Euclidean motions, an ACG of type A is a Bieberbach group); and if Γ is torsion free, we call $\Gamma\backslash V$ a *space form of type* A (every flat, complete, compact connected pseudo-Riemannian manifold is one of these, A being a suitable group of isometries; see [24, Theorem 2.4.9]).

Bieberbach proved that every Bieberbach group is a finite extension of its (free abelian) translation subgroup, that in each dimension there are only the finitely many isomorphism types of the Bieberbach group, and that isomorphic Bieberbach groups are conjugate in the affine group (see [24, §3.2]). None of these results is true of ACGs of more general type, but there are weaker analogues which do generalize, at least conjecturally. We shall explore some of these.

A long-standing conjecture [16] asserts that *every ACG is virtually polycyclic* (i.e., has a polycyclic subgroup of finite index). It has been proved for ACGs in dimension ≤ 3 [10], for ACGs of type A whenever A is an extension of the translation group by (the real points of) a reductive algebraic group of real rank at most 1 (for example, the group of affine Lorentz transformations) [13], and in some other cases (see [11], [20], [21].) In this paper we deal exclusively with virtually polycyclic ACGs. Throughout,

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