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NEGATIVE BENDING OF OPEN MANIFOLDS

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1. Introduction

In this paper we will have a new look at general existence theorems for metrics with negative Ricci curvature, which is motivated from several points of view. We will mention the most significant ones:

1) A general feeling expresses that a bending of metric yields (or preserves) negative curvature iff we bend outwards. Bending is used as an intuitive collective noun for deformations which, for instance, enlarge or shrink the metric along leaves of some foliation. (Think of the growth of spheres in hyperbolic space relative to that in Euclidean space.)

The "if" part will be supported by a simple construction of complete metrics of negative Ricci curvature $\mathrm{Ric} < 0$ on each open manifold, but we will disprove the "only" part: namely we also find bendings "inwards" for $\mathrm{Ric} < 0$, which yield existence results for closed manifolds.

2) The "classical" existence proof for metrics with negative scalar curvature $S < 0$ on closed manifolds (cf. [1], [9]) starts from some metric with negative integral scalar curvature, and the integral condition suffices to find conformal deformations to get a metric with $S < 0$.

In a coarse analogue we first construct metrics of some "huge amount" of negative Ricci curvature in one small ball, and indeed a "far-reaching" conformal diffusion yields $\mathrm{Ric} < 0$ on the whole manifold.

3) In [10] we already gave a series of existence theorems for $\mathrm{Ric} < 0$ starting with "weak" local deformations of Euclidean balls, and therefore had to cover the manifold with "compatible" balls to get $\mathrm{Ric} < 0$-metrics.

The deformation described here is "strong" in the contrasting sense just mentioned (in 2)). In particular the major technical problem of making these coverings work does not appear, and we get a short and simple argument for general existence results. But let us point out that those "weak" deformations and the subsequent covering in [10] are just the key to the borderline results; for instance, the space $\mathrm{Ric}^{<\alpha}(M)$ of metrics with Ricci

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