J. DIFFERENTIAL GEOMETRY
40 (1994) 449–459

BOUNDED 3-MANIFOLDS ADMIT NEGATIVELY CURVED METRICS WITH CONCAVE BOUNDARY

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Abstract

A metric can be constructed on any 3-manifold with nonempty boundary such that with respect to the metric the manifold has negative sectional curvature and the boundary is concave. In particular, the 3-ball admits such a metric.

Introduction

In this paper we construct a metric on any 3-manifold with boundary such that with respect to the metric the manifold has negative sectional curvature and the boundary is concave outwards. In particular, we construct such a metric on the 3-ball. This is surprising for several reasons.

Firstly, such a construction cannot be carried out in two dimensions. The Gauss-Bonnet theorem implies that the boundary of a negatively curved 2-disk is somewhere convex.

Secondly, such a metric cannot be constructed with constant negative sectional curvature. This contrasts with the recurrent theme in low-dimensional topology that negatively curved manifolds behave similarly to hyperbolic ones. Thus Thurston's geometrization conjecture states that a closed 3-manifold admitting a metric of negative sectional curvature also admits a hyperbolic metric, and Thurston has proved this for closed Haken manifolds and for bounded manifolds with totally geodesic boundary [4]. By contrast, there is no hyperbolic metric on the ball whose boundary is concave. Otherwise we could use the developing map [4] to immerse the ball into H^3 under a local isometry. An extremal point of the image would be a boundary point that could not be concave. Thus this construction can be viewed in some weak sense as giving negative evidence for the geometrization conjecture.

This metric has other strange properties. It is not induced by an immersion of the 3-ball into a complete negatively curved 3-manifold. The interior of the ball contains a null-homotopic closed geodesic. By contrast,

Received August 19, 1992. Partially supported by NSF grant DMS90-24796.