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ON THE EXISTENCE OF CONVEX HYPERSURFACES OF CONSTANT GAUSS CURVATURE IN HYPERBOLIC SPACE

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Introduction

In this paper we shall prove that a codimension-one embedded submanifold Γ of $\partial_\infty(\mathbf{H}^{n+1})$ is the asymptotic boundary of a complete embedded K-hypersurface M of a hyperbolic $(n+1)$-space $\mathbf{H}^{n+1}$ for any $K \in (-1, 0)$. By a K-hypersurface M, we mean the Gauss-Kronecker curvature of M is the constant K (recall that $K = K_{\text{ext.}} - 1$, where $K_{\text{ext.}}$ is the extrinsic curvature of M, i.e., the determinant of the second fundamental form). Our approach is to construct the desired M as the limit of K-graphs over a fixed compact domain in a horosphere for appropriate boundary data. Thus an important part of our study is an existence theory for K-hypersurfaces which are graphs over a bounded domain in a horosphere. This is accomplished by solving a Monge-Ampere equation for the Gauss curvature using the recent work of [6].

In general, a codimension-two closed submanifold Γ of $\mathbf{H}^{n+1}$ does not bound a K-hypersurface with $K > -1$. There are topological obstructions for Γ to bound a hypersurface with $K > -1$ (cf. [13]). For example, let Γ be a smooth Jordan curve in $\mathbf{H}^3$, and assume Γ bounds a surface with $K > -1$. Then the curvature of Γ never vanishes, so let $n(x)$, $x \in \Gamma$, be the unit principal normal to Γ. For $x \in \Gamma$, let $\Gamma_\epsilon(x)$ be the endpoint of the geodesic starting at x, of length ϵ, and with $n(x)$ as tangent at x. For ϵ small, Γ_ϵ is embedded and disjoint from Γ. Then the linking number $(\operatorname{mod} 2)$ of Γ and Γ_ϵ is zero [13]; so it is easy to construct Γ which bound no surface with $K > -1$.

We will see that for Γ an embedded codimension-one submanifold of a horosphere $\subset \mathbf{H}^{n+1}$, and $K \in (-1, 0)$, there exists a K-hypersurface M with boundary $\partial M = \Gamma$.

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