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KAEHLER STRUCTURES ON TORIC VARIETIES

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1. Let (X, ω) be a compact connected $2n$-dimensional manifold, and let

$$\tau: T^n \to \mathrm{Diff}(X, \omega) \tag{1.1}$$

be an effective Hamiltonian action of the standard n-torus. Let $\phi: X \to \mathbb{R}^n$ be its moment map. The image, Δ, of ϕ is a convex polytope, called the *moment polytope.* Delzant showed in [5] that the triple (X, ω, τ) is determined up to isomorphism by this polytope, and also that X has an intrinsic T^n-invariant complex structure which is compatible with ω and makes X into a toric variety. The purpose of this note is to show that is not only the symplectic geometry of X determined by Δ, but also, to a certain extent, the *Kaehler* geometry of X. By [5], Δ can be described by a set of inequalities of the form

$$\langle x, u_i\rangle \geq \lambda_i, \qquad i = 1, \cdots, d; \tag{1.2}$$

the u_i's being primitive elements of the lattice, $\mathbb{Z}^n$, and d the number of $(n-1)$-dimensional faces of Δ. Let $l_i: \mathbb{R}^n \to \mathbb{R}$ be the map

$$l_i(x) = \langle x, u_i\rangle - \lambda_i,$$

and let Δ° be the interior of Δ. Then $x \in \Delta^\circ$ if and only if $l_i(x) > 0$ for all i. Let

$$l_\infty(x) = \sum_{i=1}^{d} \langle x, u_i\rangle.$$

Our main result is the following formula for the restriction of ω to $\phi^{-1}(\Delta^\circ)$:

$$\omega = \sqrt{-1}\partial\bar{\partial}\pi^* \left(\sum_{i=1}^{d} \lambda_i(\mathrm{Log}\, l_i) + l_\infty\right). \tag{1.3}$$

This we will derive as a corollary of another result which I will now describe: By [5] there is an intrinsic involution $\gamma: X \to X$ which reverses

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