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PROOF OF THE SOUL CONJECTURE OF CHEEGER AND GROMOLL

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In this note we consider complete noncompact Riemannian manifolds M of nonnegative sectional curvature. The structure of such manifolds was discovered by Cheeger and Gromoll [2]: M contains a (not necessarily unique) totally convex and totally geodesic submanifold S without boundary, $0 \leq \dim S < \dim M$, such that M is diffeomorphic to the total space of the normal bundle of S in M. (S is called a soul of M.) In particular, if S is a single point, then M is diffeomorphic to a Euclidean space. This is the case if all sectional curvatures of M are positive, according to an earlier result of Gromoll and Meyer [3]. Cheeger and Gromoll conjectured that the same conclusion can be obtained under the weaker assumption that M contains a point where all sectional curvatures are positive. A contrapositive version of this conjecture expresses certain rigidity of manifolds with souls of positive dimension. It was verified in [2] in the cases $\dim S = 1$ and $\operatorname{codim} S = 1$, and by Marenich, Walschap, and Strake in the case $\operatorname{codim} S = 2$. Recently Marenich [4] published an argument for analytic manifolds without dimensional restrictions. (We were unable to get through that argument, containing over 50 pages of computations.)

In this note we present a short proof of the Soul Conjecture in full generality. Our argument makes use of two basic results: the Berger's version of Rauch comparison theorem [1] and the existence of distance nonincreasing retraction of M onto S due to Sharafutdinov [5].

Theorem. *Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature, let S be a soul of M, and let P: $M \to S$ be a distance nonincreasing retraction.*

(A) *For any $x \in S$, $\nu \in SN(S)$ we have*

$$P(\exp_x(t\nu)) = x \quad for\ all\ t \geq 0,$$

where $SN(S)$ denotes the unit normal bundle of S in M.

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