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TWO APPLICATIONS OF JACOBI FIELDS TO THE BILLIARD BALL PROBLEM

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Abstract

We present new proofs of two results on the billiard ball problem by Rychlik [R] and Bialy [B].

0. Introduction

We will give new proofs of two results on the billiard ball problem by Rychlik [7] and Bialy [1]. The original proofs were based on variational considerations. In our approach the variational context is absent, the dynamical system takes the center stage. We hope the simplifications provided by our method will make possible some progress on the conjectures for which these results lend partial support.

1. The dynamical system

Let us consider a convex domain Q in the plane. The billiard ball system is the flow Φ^t on $Q \times \mathbb{S}^1$ defined by the free motion of a point particle in Q, with elastic reflections at the boundary ∂Q (the angle of reflection is equal to the angle of incidence). The circle $\mathbb{S}^1$ represents unit velocities. Strictly speaking, we need to identify the velocities at the boundary according to the collision law. The flow Φ^t preserves the Liouville measure ν equal to the product of the Lebesgue measures in Q and $\mathbb{S}^1$.

Birkhoff [2] thought that this dynamical system is a very good model for Hamiltonian dynamics. In the last 30 years his belief proved to be strikingly accurate. We understand as much about the low dimensional Hamiltonian dynamics as we know about the billiard system.

The flow Φ^t has a natural section map $T : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M} = \partial Q \times [0, \pi]$. The map T describes the dynamics "from collision to collision". The space $\mathcal{M}$ is the set of unit tangent vectors attached at the boundary

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