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HOMOLOGY OF MODULI SPACES OF INSTANTONS ON ALE SPACES. I

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1. Introduction

In [13] P. B. Kronheimer and the author introduced a new class of hyper-Kähler manifolds which arise as moduli spaces of anti-self-dual connections on a certain class of 4-dimensional noncompact manifolds, the so-called ALE spaces. The ALE space is diffeomorphic to the minimal resolution of the simple singularity $\mathbb{C}^2/\Gamma$ for a finite subgroup Γ of $SU(2)$, and was constructed by P. B. Kronheimer [12]. In [17] we studied the geometry of the moduli space, and showed that, under a certain topological condition on the vector bundle (cf. (5.1)), its middle cohomology group is isomorphic to a weight space of an irreducible finite dimensional representation of a simple Lie algebra. The key geometric property of the moduli space is the existence of an S^1-action.

The aim of the present paper and its sequel is to compute the homology of the moduli spaces. The method is to use the moment map for the S^1-action as a Morse function. In this paper we treat the case where the group Γ is a cyclic group (i.e., the base ALE space is of type A_n). One of the main results in this paper is the following:

Theorem 1.1. *Let $\mathfrak{M}$ be the moduli space of anti-self-dual connections on a vector bundle over the ALE space of type A_n (see §2 for more precise definitions). Suppose that the hyper-Kähler metric on $\mathfrak{M}$ is complete. Then $\mathfrak{M}$ has a nondegenerate Morse function F which has only critical points of even index. In particular, the homology of $\mathfrak{M}$ has no torsion and vanishes in odd degrees, and every component of $\mathfrak{M}$ is simply connected.*

In fact, we can say more: we have a precise description of the critical points and an algorithm to compute the Betti numbers (Theorem 3.2, equation (3.4), Proposition 4.3).

It was shown that the homology group of the moduli space is isomorphic to that of Spaltenstein's variety in [17, 8.7] when the ALE space is

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