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IRREDUCIBILITY OF MODULI OF RANK-2 VECTOR BUNDLES ON ALGEBRAIC SURFACES

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Let X be a smooth algebraic surface over $\mathbb{C}$, let I be a fixed line bundle on X, and let H be a very ample line bundle on X. We recall that a sheaf E is H-stable (resp. H-semistable) if it is coherent, torsion free and so that for any proper subsheaf $F \subset E$, we have $p_F \prec p_E$ (resp. $p_F \preceq p_E$), where $p_E = (1/\operatorname{rank} E)\chi_E$ and $\chi_E(n) = \chi(E \otimes H^{\otimes n})$. Here by $p_F \prec p_E$ we mean $p_F(n) < p_E(n)$ for all sufficiently large n. There is a coarse moduli space $\mathfrak{M}_X^{d,I}$ parameterizing all rank 2 H-semistable sheaves E with $\det E = I$ and $c_2(E) = d$ (modulo a certain equivalence). $\mathfrak{M}_X^{d,I}$ is a projective scheme [8]. For small d, $\mathfrak{M}_X^{d,I}$ can have rather wild behavior, e.g., the dimension of $\mathfrak{M}_X^{d,I}$ may be larger than expected [9]. However, S. Donaldson [4], later generalized by R. Friedman [6] and K. Zhu [34] showed that for large d, every component of $\mathfrak{M}_X^{d,I}$ is reduced and has the expected dimension. $\mathfrak{M}_X^{d,I}$ is also normal [20] for $d \gg 0$.

Our purposes of this paper is twofold. The first is to develop a method of studying $\mathfrak{M}_X^{d,I}$ by degeneration. The second is to use this method to prove

Main Theorem. *Let X be any smooth algebraic surface over $\mathbb{C}$, and I be a fixed ample divisor. Then there is a constant A depending on (X, H, I) such that whenever $d \geq A$, then $\mathfrak{M}_X^{d,I}$ is irreducible.*

The proof of the theorem is based on the following well-known observation: Let A be large so that for $d \geq A$, $\mathfrak{M}_X^{d,I}$ is smooth at a dense subset. Take $\mathbf{M} \subseteq \mathfrak{M}_X^{d,I}$ be any irreducible component and take $E \in \mathbf{M}$ be a smooth point. Let $\mathbb{C}_x$ be the skyscraper sheaf over $x \in X$ and let $E \to \mathbb{C}_x$ be a general surjective homomorphism. The kernel E' of $E \to \mathbb{C}_x$ is a stable sheaf with $c_2(E') = d + 1$ and $\operatorname{Ext}^2(E', E')^0 = \{0\}$. Thus E' belongs to a unique irreducible component of $\mathfrak{M}_X^{d+1,I}$. Now if we let $\Lambda(d)$ be the set of irreducible components of $\mathfrak{M}_X^{d,I}$, then this construction

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