J. DIFFERENTIAL GEOMETRY
39 (1994) 659-680

CONJUGACY AND RIGIDITY FOR MANIFOLDS WITH A PARALLEL VECTOR FIELD

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Abstract

The main theorem in this paper is that any compact Riemannian manifold with geodesic flow isomorphic to the geodesic flow of a local Riemannian product $M = (X \times R)/\Gamma$ is isometric to M.

1. Introduction

In this paper we consider the question: Which compact Riemannian manifolds M are determined uniquely by their geodesic flows? To formulate this precisely we need a few definitions. If M and N are complete Riemannian manifolds, then their geodesic flows are C^0 conjugate if there is a homeomorphism $F: SM \to SN$ from the unit sphere bundle SM to the unit sphere bundle SN which commutes with the geodesic flows: $F \circ g_M^t = g_N^t \circ F$ for all $t \in R$ where g^t is the geodesic flow after time t. If $0 \le r \le \infty$, and F can be chosen to be a C^r diffeomorphism, then M and N have C^r conjugate geodesic flows. A complete Riemannian manifold M is C^r conjugacy rigid if any Riemannian manifold N whose geodesic flow is C^r conjugate to the geodesic flow of M is isometric to M. A more precise formulation of our question then is: Which compact Riemannian manifolds M are C^r conjugacy rigid?

It was pointed out by Weinstein (see [2, §4F]) that the geodesic flow of a Zoll surface is C^∞ conjugate to the geodesic flow of a round sphere. Using a variation of this idea we show in §6 that on any smooth manifold there are infinite-dimensional families of pairwise nonisometric metrics with mutually C^∞ conjugate geodesic flows. In particular, any Riemannian manifold containing an open subset isometric to a neighborhood of an equator $S^{n-1}(1) \subset S^n(1)$ is not conjugacy rigid.

On the other hand, surfaces of nonpositive curvature are C^0 conjugacy rigid (see [4] for the C^1 case, and [6] for the C^0 case). When both M and

Received October 6, 1992, and, in revised form, August 9, 1993. The first author was supported in part by NSF grant #DMS90-01707 and #D MS85-05550 while at MSRI. The second author was supported by an NSF Postdoctoral Fellowship.