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THIN POSITION AND HEEGAARD SPLITTINGS OF THE 3-SPHERE

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We present here a simplified proof of the theorem, originally due to Waldhausen [7], that a Heegaard splitting of S^3 is determined solely by its genus. The proof combines Gabai's powerful idea of "thin position" [2] with Johannson's [4] elementary proof of Haken's theorem [3] (Heegaard splittings of reducible 3-manifolds are reducible). In §3.1, 3.2 & 3.8 we borrow from Otal [6] the idea of viewing the Heegaard splitting as a graph in 3-space in which we seek an unknotted cycle.

Along the way we show also that Heegaard splittings of boundary reducible 3-manifolds are boundary reducible [1, 1.2], obtain some (apparently new) characterizations of graphs in 3-space with boundary-reducible complement, and recapture a critical lemma of [5]. We are indebted to Erhard Luft for pointing out a gap in the original argument.

1. Heegaard splittings: a brief review

1.1. All surfaces and 3-manifolds will be compact and orientable. A *compression body* H is constructed by adding 2-handles to a (surface) $\times$ 1 along a collection of disjoint simple closed curves on (surface) $\times\{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component (surface) $\times\{1\}$ of ∂H is denoted $\partial_+ H$ and the surface $\partial H - \partial_+ H$, which may or may not be connected, is denoted $\partial_- H$ (Figure 1a, next page). If $\partial_- H = \emptyset$, then H is a *handlebody*. If $H = \partial_+ H \times 1$, H is called a *trivial* compression body. A *spine* for H is a properly imbedded 1-complex Q such that H collapses to $Q \cup \partial_- H$ (Figure 1b).

1.2. Spines are not unique, but can be altered by *edge slides*, as follows: Choose an edge e in Q and let $\overline{Q}$ be the graph $Q - e$. Let $\overline{H}$ denote a regular neighborhood of $\partial_- H \cup \overline{Q}$. Then H is the union of $\overline{H}$ and a 1-handle h attached to $\partial_+ \overline{H}$. The core of h is the edge e, with its ends

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