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A SURFACE WITH POSITIVE CURVATURE AND POSITIVE TOPOLOGICAL ENTROPY

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Abstract

We construct explicit examples of closed surfaces with positive curvature whose geodesic flow has positive topological entropy. It follows that these surfaces have infinitely many hyperbolic closed geodesics.

1. Introduction

We begin by posing the following general question:

Question. *Let M be a C^∞ closed orientable surface and g a smooth Riemannian metric with positive Gaussian curvature. Can the geodesic flow for g have a complicated dynamical behavior?*

The Gauss-Bonnet Theorem tells us that a positively curved surface must be topologically a sphere. The most common examples are the round sphere (and other surfaces of revolution) and the tri-axial ellipsoid. Both of these examples possess simple dynamics (i.e., their geodesic flows are not ergodic and they have zero entropies.) One might think that the simple topology of the sphere could be an obstruction for the geodesic flow of g to have complicated dynamics. This is not the case. Donnay [7] and Burns and Gerber [3] have constructed smooth (and real analytic) metrics on the sphere whose geodesic flows are Bernoulli. Katok [12] has shown that the simple topology is not an obstruction for a map to possess complicated dynamical behavior by constructing Bernoulli diffeomorphisms of the 2-disk.

Donnay, Burns, and Gerber construct their metrics by starting with a thrice punctured sphere and considering its complete hyperbolic metric. They then alter the metric far off into the cusps by cutting off the remainder of the cusps and glueing in reflecting caps. The geodesics leave the reflecting caps focused as they entered, and the cone family can be controlled in the caps. It is clear that these examples have "mostly" negative

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