J. DIFFERENTIAL GEOMETRY
39 (1994) 139-172

SUBVARIETIES OF GENERAL HYPERSURFACES IN PROJECTIVE SPACE

GENG XU

0. Introduction

We are interested in the following question: If C is an irreducible curve (possibly singular) on a generic surface of degree d in a projective 3-space $\mathbf{P}^3$, can the geometric genus of C (the genus of the desingularization of C) be bound from below in terms of d? Bogomolov and Mumford [14] have proved that there is a rational curve and a family of elliptic curves on every K-3 surface. Since a smooth quartic surface in $\mathbf{P}^3$ is a K-3 surface, there are rational and elliptic curves on a generic quartic surface in $\mathbf{P}^3$. On the other hand, Harris conjectured that on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$ there are neither rational nor elliptic curves.

Now let C be a curve on a surface S of degree d in $\mathbf{P}^3$. By the Noether-Lefschetz Theorem, if $d \geq 4$ and S is generic, then C must be a complete intersection of S with another surface S_1 of degree k. In this case we say that C is a type (d, k) curve on S. Clemens [4] has proved that there is no type (d, k) curve with geometric genus $g \leq \frac{1}{2}dk(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$; in particular, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-5)$ on a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$.

Our first main result is the following.

Theorem 1. *On a generic surface of degree $d \geq 5$ in $\mathbf{P}^3$, there is no curve with geometric genus $g \leq \frac{1}{2}d(d-3)-3$, and this bound is sharp. Moreover this sharp bound can be achieved only by a tritangent hyperplane section if $d \geq 6$.*

We immediately conclude that the above conjecture of Harris is true. Meanwhile it is not hard to see that for a generic surface S of degree d in $\mathbf{P}^3$, there is a tritangent hyperplane H and thus $C = H \cap S$ has three double points. Since $\pi(C) = \frac{1}{2}(C\cdot C + K_S \cdot C) + 1 = \frac{1}{2}d(d-3)+1$, and an ordinary double point drops the genus of a curve by 1, the above bound is sharp.

Received January 11, 1993. Partially supported by Sloan Doctoral Dissertation Fellowship.