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SURFACES WITH GENERALIZED SECOND FUNDAMENTAL FORM IN L^2 ARE LIPSCHITZ MANIFOLDS

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Abstract

This paper focuses in the relationship between the class of surfaces with second fundamental form in L^2 and the class of Lipschitz surfaces (i.e., surfaces that are locally homeomorphic via a bilipschitz map to a flat disc). In particular we prove that graphs of $W^{2,2}(\mathbf{R}^2)$ functions are Lipschitz surfaces.

Introduction

For functions u, defined on a domain $\Omega \subset \mathbf{R}^2$, having locally square integrable partial derivatives up to order 2 (in the generalized sense), the Sobolev embedding theorems guarantee that u is locally Hölder continuous with any exponent $\alpha < 1$, and also that the gradient Du is locally in L^p for every $p < \infty$. There are, of course, examples illustrating that such u may not be locally Lipschitz—that is, Du need not be locally bounded in Ω. Since it gives some important insight into the nature of the singularities of general $W^{2,2}$ functions, we discuss a couple of particular examples in some detail.

Example 1. Let D be the disc of radius $\frac{1}{2}$ in $\mathbf{R}^2$, and let $u : D \to \mathbf{R}$ be defined by $u(x, y) = x \log|\log r|$, where $r = \sqrt{x^2 + y^2}$. Direct computation shows that the Hessian D^2u is in $L^2(D)$; in fact $|D^2u| \leq Cr^{-1}|\log r|^{-1}$. On the other hand, Du is evidently unbounded on approach to the origin, in fact

$$u_x = \log|\log r| + O(|\log r|^{-1}) \quad \text{and} \quad u_y = O(|\log r|^{-1}) \quad \text{as } r \downarrow 0.$$

One can easily check that while Du has a singularity at 0, the unit normal $\nu = (1 + |du|^2)^{-1/2}(-Du, 1)$ has limit $-e_1 = (-1, 0, 0)$ as $r \downarrow 0$ and the graph of u is a C^1 surface embedded in $\mathbf{R}^3$ with tangent plane normal

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