J. DIFFERENTIAL GEOMETRY
39 (1994) 51-55

LOCAL PROPERTIES OF FAMILIES OF PLANE CURVES

ROBERT TREGER

Introduction

Let $\mathbf{P}^N$ be the projective space parametrizing all projective plane curves of degree n $(N = n(n+3)/2)$. For $d \geq 1$, we let $\Sigma_{n,d} \subset \mathbf{P}^N \times \mathrm{Sym}^d(\mathbf{P}^2)$ be the closure of the locus of pairs $(E, \Sigma_{i=1}^d P_i)$, where E is an irreducible nodal curve and $P_1, \cdots, P_d$ are its nodes. The purpose of this paper is to prove the following theorem.

Theorem. *The variety $\Sigma_{n,d}$ is unibranch everywhere.*

The variety $\Sigma_{n,d}$ plays an important role in the study of the family of irreducible plane curves of degree n with d nodes and no other singularities as well as the locus $V(n, g) \subset \mathbf{P}^N$ of reduced and irreducible curves of genus g, where $g = (n-1)(n-2)/2 - d$. We mention two corollaries.

Corollary 1 (Harris [5]). *The variety $\overline{V(n, g)} \subset \mathbf{P}^N$ is irreducible.*

Corollary 2. *The locus $V(n, g)$ is unibranch everywhere.*

It is well known that $\overline{V(n, g)}$ is not unibranch everywhere [3], [5, §1], [6, Lecture 3], [10, §11]. We now prove the corollaries. Recall a result of Arbarello and Cornalba [1] and Zariski [13]: *the general members of $V(n, g)$ have $d = (n-1)(n-2)/2 - g$ nodes and no other singularities.* It follows that the projection of $\Sigma_{n,d}$ to $\mathbf{P}^N$ coincides with $\overline{V(n, g)}$. Every component of $\Sigma_{n,d}$ contains a pair of the form $(\Sigma_{r=1}^n L_r, dP)$, where the lines L_r $(1 \leq r \leq n)$ meet only at P, and by the deformation theory, $\Sigma_{n,d}$ contains all such pairs [6, Lecture 3, §2], [10, §11]. It is clear that these pairs form an irreducible family. Hence $\Sigma_{n,d}$ is irreducible by our theorem. It follows that $\overline{V(n, g)}$ is also irreducible.

We now prove Corollary 2. Let C be an arbitrary member of $V(n, g)$. For a point $P \in C$, we set $\delta_P = \dim_{\mathbf{C}} \widetilde{O}_P/O_P$, where O_P is the local ring of C at P, and $\widetilde{O}_P$ its normalization. By the genus formula, $\Sigma_{Q \in C} \delta_Q = d$ [7, Theorem 2]. Therefore if a nodal member of $V(n, g)$ specializes to C, then exactly δ_P of its nodes specialize to $P \in C$ [12, §3.4]. Hence C

Received January 16, 1990, and, in revised form, April 10, 1992.