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NORMAL GENERATION OF VECTOR BUNDLES OVER A CURVE

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Introduction

Many algebro-geometric calculations simplify when a particular multiplication map of global sections on vector bundles surjects. For example, a line bundle over a smooth variety X is normally generated iff L embeds X as a projectively normal variety. Equivalently, L is normally generated iff L is ample and for all $n \geq 1$ we have surjectivity for the multiplication map

$$S^n(H^0(X, L)) \to H^0(X, L^{\otimes n}).$$

This characterization means normal generation allows us to calculate dimensions of the spaces of quadrics, cubics, and so forth which vanish on X, using Riemann-Roch. So we want optimal numerical conditions forcing normal generation. Mumford [30] shows that if L_1 and L_2 are line bundles over a smooth curve C of genus g with $\deg(L_1) \geq 2g$ and $\deg(L_2) \geq 2g+1$, we have surjectivity of the multiplication map

$$\tau: H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2).$$

So L is normally generated whenever $\deg(L) \geq 2g+1$, which was first discovered by Castelnuovo [5], then rediscovered by Mattuck [27], and again by Mumford [30]. Examples show this result is optimal.

Recent work generalizes the " $2g+1$ " theorem to higher syzygies. We recall notation introduced by Green and Lazarsfeld [16]. L has property N_0 iff L is normally generated. L has property N_1 iff L is normally presented, meaning L has property N_0 and the homogeneous ideal is generated by quadrics. L has property N_2 iff L has property N_1 and the relations among the quadrics are generated by the linear relations. L has property N_3 iff L has property N_2 and the relations among the relations are generated by the linear relations. And so on. Still working over a smooth curve C, Fujita [9] and Saint-Donat [36] built upon work of Mumford and showed L is normally presented provided $\deg(L) \geq 2g+2$.

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