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THE FRÖHLICHER SPECTRAL SEQUENCE ON A TWISTOR SPACE

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Abstract

Precise results are obtained about the degeneration of the Fröhlicher spectral sequence on the twistor space of a compact self-dual 4-manifold and several examples are studied. One of these shows that, for compact complex 3-manifolds, the property of *nondegeneration* of Fröhlicher is *unstable* under deformations of complex structure. Another consequence of the analysis is the discovery of a period mapping for (Riemannian) conformal structures on a compact 4-manifold.

1. Introduction

Associated to any compact self-dual 4-manifold M is a compact complex three-dimensional manifold Z known as its twistor space [1], [20]. Twistor spaces provide a source of interesting complex three-manifolds (cf. [16]). The purpose of this article is to investigate the Fröhlicher spectral sequence [8]

$$E_1^{p,q} = H^q(Z, \Omega^p) \Rightarrow H^{p+q}(Z, \mathbf{C}),$$

where Ω^p denotes the sheaf of holomorphic p-forms on Z. The Penrose transform [2], [3], [4], [6], [11] interprets the Dolbeault cohomology $H^q(Z, \Omega^p)$ in terms of differential equations on M. In this way, the Fröhlicher spectral sequence has differential-geometric consequences on M, and vice versa.

We shall explain this interpretation and its consequences. For example, we shall show that the spectral sequence is *degenerate* (i.e., $E_1 = E_\infty$) if and only if a certain conformally invariant system of linear differential equations has only constant solutions. The classical case in which $E_1 = E_\infty$ is when Z admits a Kähler metric. Hitchin [12] has shown that there are only two such twistor spaces, namely $\mathbf{CP}_3$ and the space of flags in $\mathbf{C}^3$. However, we shall construct other twistor spaces with $E_1 = E_\infty$. We shall

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