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DISCRETE PARABOLIC GROUPS

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0. Introduction

Let X be a complete simply connected Riemannian manifold of pinched negative curvature (i.e., all the sectional curvatures lie between two negative constants). The main aim of this paper is to show that any discrete group of isometries of X fixing some ideal points is finitely generated (Theorem 4.1). The only interesting case is that of a discrete parabolic group (preserving setwise some horosphere in X). In this case, by applying the Margulis Lemma (2.3), it follows that a discrete parabolic group contains a nilpotent subgroup of finite index. We may identify this subgroup as the group generated by those elements having "small rotational part". In fact, the notion of the rotational part of a parabolic isometry will be one of the main ingredients of the proof of the main theorem (see §3).

Conversely, it is well known that any (virtually) nilpotent discrete group of isometries must be "elementary". In particular, some finite-index subgroup must fix an ideal point. Thus, all discrete nilpotent groups are finitely generated. This rules out the possibility of groups such as the rational numbers occurring as discrete groups. (Note that there is no purely topological obstruction to this—see the end of §4.)

I suspect that one should be able to strengthen the conclusion of the main theorem, to show that the quotient space of a discrete parabolic group is always topologically finite, i.e., homeomorphic (or diffeomorphic) to the interior of a compact orbifold. However, I do not have a proof of this.

The case where X has constant negative curvature is a consequence of the Bieberbach Theorem (§1). Proofs of the Bieberbach theorem usually proceed by an induction on dimension, and so an argument along these lines would make essential use of the existence of totally geodesic subspaces. Thus, for the variable curvature case, we will need another approach.

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