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CLOSED HYPERBOLIC 3-MANIFOLDS WHOSE CLOSED GEODESICS ALL ARE SIMPLE

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1. Introduction

Let M be a complete orientable hyperbolic n-manifold of constant curvature -1 and finite volume (in the sequel this will be abbreviated to simply "hyperbolic n-manifold").

A closed geodesic in M is *simple* if it has no self-intersections, and *nonsimple* otherwise. When $n = 2$, there are always nonsimple closed geodesics. Indeed much effort has been made in the case $n = 2$ to algorithmically describe simple geodesics; see for example [3], [4]. Moreover there are hyperbolic 2-manifolds where no closed geodesic is simple. For example, since the Teichmüller space of a twice-punctured disc is a point, it has a unique hyperbolic structure, and it is not difficult to see that all closed geodesics in this case are nonsimple. Here we prove the following result:

Theorem 1. *There exist infinitely many noncommensurable closed hyperbolic* 3*-manifolds all of whose closed geodesics are simple.*

Whether such examples exist has been frequently asked. Our examples are arithmetic, and are constructed via the theory of quaternion algebras. More precisely, suppose Γ is a torsion-free arithmetic Kleinian group which is derived from a quaternion algebra B over a number field, in the sense of §2.2. If B is a division algebra, then $M = \mathbf{H}^3/\Gamma$ is a closed hyperbolic 3-manifold. We show in Proposition 5 that if M has a nonsimple closed geodesic, then B must have a Hilbert symbol $\{a, b\}$ of a particular kind. In §4 we show that there are infinitely many nonisomorphic B which do not have Hilbert symbols of the above kind; this implies Theorem 1.

While Hilbert symbols thus provide a nontrivial obstruction to the existence of nonsimple closed geodesics, we do not know if this is the only obstruction for arithmetic M as above. We should also mention that the

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