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LOOP SPACES AS COMPLEX MANIFOLDS

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1. Introduction

Given a topological space M, its loop space consists of mappings of the circle S^1 into M. Depending on what conditions we impose on the mappings we get several loop spaces associated with M. If M has more structure than just topological, the loop spaces tend to inherit this structure: for example if M is a (Riemannian) manifold, the space of smooth loops is also a (Riemannian) manifold, albeit infinite dimensional. There is nothing surprising about this. In some cases, however, it happens that the *interaction* of the structures of M and S^1 gives rise to a structure on a loop space. For example with G a compact Lie group, the space of smooth loops in G modulo the action of G is a complex manifold (see [20]). Similarly, the manifold $\mathrm{Diff}\, S^1/S^1$ is also a complex manifold (see [3], [10]). Here $\mathrm{Diff}\, S^1$ stands for the space of smooth, orientation preserving diffeomorphisms of the circle, hence can be thought of as a space of embedded smooth loops in S^1.

More recently J. Brylinski observed that the manifold of smooth, oriented, unparametrized knots in an oriented Riemannian manifold (M, g) of dimension 3 also has a complex structure; see [4]. We shall now describe this complex structure, which naturally lives on the space of immersed rather than embedded loops (knots).

Thus, let $\mathfrak{M}$ denote the set of equivalence classes of smooth (meaning C^∞) immersions $f: S^1 \to M$. Two immersions $f_1, f_2: S^1 \to M$ are equivalent if $f_1 = f_2 \circ \varphi$, with φ an orientation preserving diffeomorphism of S^1. Elements of $\mathfrak{M}$ are called immersed loops. First we endow $\mathfrak{M}$ with a topology as follows. Fix an immersed loop $\Gamma \in \mathfrak{M}$ represented by $f: S^1 \to M$. Let $\nu \to S^1$ denote the normal bundle of f:

$$\nu = \bigcup_{t \in S^1} \{v \in T_{f(t)}M : v \perp f_* T_t S^1\},$$

and exp the (partially defined) exponential map $\nu \to M$. ν inherits a Riemannian metric and a connection from TM, and so it makes sense

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