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THE MOMENT MAP AND LINE BUNDLES OVER PRESYMPLECTIC TORIC MANIFOLDS

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Abstract

We apply symplectic methods in studying smooth toric varieties with a closed, invariant 2-form ω that may have degeneracies. Consider the push-forward of Liouville measure by the moment map. We show that it is a "twisted polytope" in $\mathfrak{t}^*$ which is determined by the winding numbers of a map $S^{n-1} \to \mathfrak{t}^*$ around points in $\mathfrak{t}^*$. The index of an equivariant, holomorphic line-bundle with curvature ω is a virtual T-representation which can easily be read from this "twisted polytope".

1. Introduction

A *symplectic manifold* is a smooth manifold M with a closed 2-form ω which is everywhere nondegenerate. Let T be a compact torus which acts effectively, preserving ω. A *moment map* for (M, T, ω) is a map $\Phi: M \to \mathfrak{t}^*$ such that $\langle d\Phi, \xi\rangle = -i(\xi_M)\omega$ for every $\xi \in \mathfrak{t}$, where ξ_M denotes the corresponding vector field on M. By the Atiyah-Guillemin-Sternberg convexity theorem [1], [12], the image of the moment map is a convex polytope Δ. For an excellent introduction to this subject, see [3].

If (M, T, ω) admits a moment map, then the dimension of T cannot exceed half of the dimension of M. If $\dim T = \frac{1}{2}\dim M$, then the action is *completely integrable.* Delzant [5] classifies these spaces; the polytope Δ determines (M, T, ω) up to equivariant symplectomorphism. Moreover, he shows that (M, T) is equivariantly diffeomorphic to a *toric manifold*, i.e., a smooth toric variety.

In particular, M admits a complex structure such that T acts holomorphically. Let L be an equivariant holomorphic line bundle over M with curvature ω, where ω is the imaginary part of a Kähler form on M. Denote the sheaf of holomorphic sections of L by $\mathcal{O}_L$. Then $H^i(M, \mathcal{O}_L)$ is a representation of T. Danilov [4] shows that the weights which occur in $H^0(M, \mathcal{O}_L)$ are exactly the lattice points in Δ (with multiplicity one), whereas $H^i(M, \mathcal{O}_L) = 0$ for $i > 0$.

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