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CONVERGENCE OF THE ALLEN-CAHN EQUATION TO BRAKKE'S MOTION BY MEAN CURVATURE

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Abstract

The equation $\partial u^\varepsilon/\partial t = \Delta u^\varepsilon - (1/\varepsilon^2)f(u^\varepsilon)$ was introduced by Allen and Cahn to model the evolution of phase boundaries driven by isotropic surface tension. Here $f = F'$ and F is a potential with two equal wells. We prove that the measures $d\mu_t^\varepsilon \equiv ((\varepsilon/2)|Du^\varepsilon|^2 + (1/\varepsilon)F(u^\varepsilon))\,dx$ converge to Brakke's motion of varifolds by mean curvature. In consequence, the limiting interface is a closed set of finite $\mathscr{H}^{n-1}$-measure for each $t \geq 0$ and of finite $\mathscr{H}^n$-measure in spacetime. In particular the limiting interface is a "thin" subset of the level-set flow (which can fatten up) and satisfies the maximum principle when tested against smooth, disjoint surfaces moving by mean curvature. The main tools are Huisken's monotonicity formula, Evans-Spruck's lower density bound and equipartition of energy. In addition, drawing on Brakke's regularity theory, there is almost-everywhere regularity for generic (i.e., nonfattening) initial condition.

Introduction

The equation

$$(*) \qquad \frac{\partial}{\partial t}u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2}f(u^\varepsilon)$$

was introduced by Allen and Cahn in 1979 to model the motion of phase boundaries by surface tension [2]. Here f is the derivative of a potential F with two wells of equal depth at $u = \pm 1$. The equation is the gradient flow of

$$M^\varepsilon[u] = \int \frac{\varepsilon}{2}|Du|^2 + \frac{1}{\varepsilon}F(u)\,dx,$$

sped up by the factor $1/\varepsilon$.

The effect of $-(1/\varepsilon)^2 f$ is to force u^ε to approximate a characteristic function, with a transition layer of width approximately ε and slope approximately C/ε. Heuristically, the interface should move by mean curvature in the limit.

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