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PINCHING AND CONCORDANCE THEORY

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Abstract

It is known that a complete simply connected Riemannian manifold M whose sectional curvature $\sec(M)$ satisfies $1/4 < \sec(M) \le 1$ is homeomorphic to a sphere. Beyond that, the diffeomorphism type of M is subject to a symmetry condition formulated in this paper. Methods from concordance theory and algebraic K-theory show that many exotic spheres do not satisfy the condition.

0. Introduction

The sphere theorem of Rauch [20], Berger [1], and Klingenberg [18] states that a complete simply connected Riemannian manifold M whose sectional curvature $\sec(M)$ satisfies $1/4 < \sec(M) \le 1$ everywhere is homeomorphic to a sphere. Grove and Shiohama [12] have obtained the same conclusion from a weaker hypothesis on the Riemannian metric (details below). Should it not be possible to keep the original hypothesis and get a stronger conclusion? In connection with this question, the notion of *Morse perfection* seems to be useful.

Let N^n be a closed smooth manifold and let $W(N)$ be the set of all smooth Morse functions on N having only two critical points (necessarily of index 0 and n). Of course, this may well be empty. In any case, $Z/2$ acts freely on $W(N)$ by $f \mapsto -f$ (for $f \in W(N)$).

0.1 Definition. The *Morse perfection* of N is $\ge k$ if there exists a smooth $Z/2$-map $q: S^k \to W(N)$ where $Z/2$ acts on S^k by the antipodal action. (By definition, q is *smooth* if its adjoint $q^{\#}: S^k \times N \to \mathbb{R}$ is smooth.)

First examples:

(i) Any N has Morse perfection ≥ -1.

(ii) The Morse perfection of N^n is ≥ 0 if and only if $W(N) \ne \varnothing$, and in this case N is homeomorphic to S^n.

(iii) The standard sphere S^n has Morse perfection $\ge n$. (Define q by $q^{\#}(z, y) = \langle z, y\rangle$ for $z, y \in S^n$, using the Euclidean scalar product in $\mathbb{R}^{n+1} \supset S^n$.)

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