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THE BRUNN-MINKOWSKI-FIREY THEORY I: MIXED VOLUMES AND THE MINKOWSKI PROBLEM

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The Brunn-Minkowski theory is the heart of quantitative convexity. It had its origins in Minkowski's joining his notion of mixed volumes with the Brunn-Minkowski inequality. One of Minkowski's major contributions to the theory was to show how this theory could be developed from a few basic concepts: support functions, Minkowski combinations, and mixed volumes. Thirty years ago, Firey [8] (see Burago and Zalgaller [4, §24.6]) extended the notion of a Minkowski combination, and introduced, for each real $p \geq 1$, what he called p-sums.

It is the aim of this series of articles to show that these Firey combinations lead to a Brunn-Minkowski theory for each $p \geq 1$.

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean n-space, $\mathbb{R}^n$. Let $\mathscr{K}_o^n$ denote the set of convex bodies containing the origin in their interiors. For $K \in \mathscr{K}^n$, let $h_K = h(K, \cdot): S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h_K(u) = h(K, u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the standard inner product in $\mathbb{R}^n$. The set $\mathscr{K}^n$ will be viewed as equipped with the usual Hausdorff metric, d, defined by $d(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere, $C(S^{n-1})$.

For $K, L \in \mathscr{K}^n$, and $\alpha, \beta \geq 0$ (not both zero), the Minkowski linear combination $\alpha K + \beta L \in \mathscr{K}^n$ is defined by

$$h(\alpha K + \beta L, \cdot) = \alpha h(K, \cdot) + \beta h(L, \cdot).$$

Firey [8] introduced, for each real $p \geq 1$, new linear combinations of convex bodies: For $K, L \in \mathscr{K}_o^n$, and $\alpha, \beta \geq 0$ (not both zero), the Firey combination $\alpha \cdot K +_p \beta \cdot L \in \mathscr{K}_o^n$ can be defined by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

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