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ON THE MODULI SPACE OF SU(n) MONOPOLES AND HOLOMORPHIC MAPS TO FLAG MANIFOLDS

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Abstract

In this paper, motivated by questions in mathematical physics, we study the geometry of the components in the spaces of based holomorphic maps from the Riemann sphere to complex flag manifolds, which we denote by $\mathrm{Rat}_C(\mathbb{F}(J))$. We decompose these spaces into smooth, in fact, complex strata each having a complex normal bundle. Using a modification of this filtration we study the forgetful map $\iota_*: H_*(\mathrm{Rat}_C(\mathbb{F}(J)) \to H_*(\Omega^2\mathbb{F}(J))$ and prove an Atiyah-Jones type stability theorem. We also use the filtrations to determine the basic groups $H_*(\mathrm{Rat}_{\bar{1}}(\mathbb{F}(J)); \mathbb{Z})$ and show that ι_* has a nontrivial kernel for general flag manifolds.

1. Introduction

Let $S^2 = \mathbb{CP}(1)$ denote the Riemann sphere, and $\mathbb{F}(J)$ the flag manifold of all sequences of complex j_i-dimensional planes through the origin in $\mathbb{C}^n$ for a fixed sequence $J = (0 < j_1 < \cdots < j_m < n)$. Thus, a point in $\mathbb{F}(J)$ is given by the flag

$$\{0\} \subset \mathbb{V}^{j_1} \subset \cdots \subset \mathbb{V}^{j_m} \subset \mathbb{C}^n,$$

where $\mathbb{V}^{j_i}$ is a complex j_i-dimensional plane through the origin. We denote the space of based holomorphic maps of the Riemann sphere to the complex flag manifold $\mathbb{F}(J)$ by $\mathrm{Rat}(\mathbb{F}(J))$, and denote its natural inclusion into $\Omega^2\mathbb{F}(J)$ given by forgetting the complex structure by

$$(1.1) \qquad \iota(J): \mathrm{Rat}(\mathbb{F}(J)) \to \Omega^2\mathbb{F}(J).$$

The $\mathrm{Rat}(\mathbb{F}(J))$ spaces occur naturally in at least two distinct contexts: first, as moduli spaces of $\mathrm{SU}(n)$-monopoles [8], [9], [13] and second, when $m = 1$, as moduli spaces of linear control equations. Additionally, it had originally been hoped when we started this work that understanding these spaces would lead to a better understanding of the instanton moduli spaces. Indeed, our recent proof of the Atiyah-Jones conjecture [2] is

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