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ON TORI EMBEDDED IN FOUR-MANIFOLDS

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1. Introduction

The genus of a smooth curve C inside a complex surface S is related to the self-intersection of C via the adjunction formula:

$$C.C + k_S.C = 2\,\mathrm{genus}(C) - 2,$$

where k_S is the canonical class of S. If S is a minimal irrational surface then $k_S.C \geq 0$. Therefore smooth complex curves inside minimal irrational surfaces satisfy

$$(*) \qquad C.C \leq 2\,\mathrm{genus}(C) - 2.$$

Moreover, there is a long-standing conjecture (originally stated by René Thom for the projective plane) which says that if $F \hookrightarrow S$ is a smoothly embedded Riemann surface homologous to C, then $\mathrm{genus}(F) \geq \mathrm{genus}(C)$. So it is natural to conjecture that $(*)$ is satisfied by smoothly embedded Riemann surfaces $F \hookrightarrow S$. When S is a Dolgachev surface and F is a 2-sphere, this has been verified by Friedman and Morgan [6], [7] using the Γ-invariant introduced by Donaldson in [4]. Also, Morgan, Mrówka and Ruberman [9] proved that if M is a closed, oriented, simply connected, smooth 4-manifold whose intersection form has positive part $b_2^+ > 1$ odd and M has some nonzero Donaldson invariant, then the following hold:

(1) if $S^2 \hookrightarrow M$ is a smoothly embedded 2-sphere representing a nontrivial homology class in M, then $S^2.S^2 < 0$ ([8]),

(2) if $T^2 \hookrightarrow M$ is a smoothly embedded 2-torus representing a nontrivial homology class, then $T^2.T^2 < 2$ ([10]).

By a result of Donaldson every smooth simply connected complex projective surface has nonvanishing Donaldson polynomial invariants; hence (1) and (2) give slightly weaker inequalities than $(*)$ for smooth projective surfaces.

To prove (2) the idea is to pull apart the 4-manifold along the boundary Y of a tubular neighborhood of an embedded sphere (or torus) violating

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