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ETERNAL SOLUTIONS TO THE RICCI FLOW

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1. The result

We consider solutions to the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

on a manifold X of dimension n. We say the solution is eternal if it is defined for all time $-\infty < t < \infty$. We are interested in solutions which are complete (which is a way of saying they are also defined for "all" of space) and which have their Riemannian curvature uniformly bounded for all space and time. This is a serious restriction; by the work of W. X. Shi [2] we know then that all the covariant derivatives of the curvature are bounded.

Examples of eternal solutions which are complete with bounded curvature are provided by solitons. These are solutions which move under a one-parameter family of diffeomorphisms. If this comes from exponentiating a vector field V_i, then we have a soliton when

$$D_i V_j + D_j V_i = 2R_{ij},$$

since the metric changes by its Lie derivative along the vector field. When the vector field is the gradient of a function we say we have a gradient soliton. If $V_i = D_i f$, the equation for a gradient soliton is

$$D_i D_j f = R_{ij},$$

so the Ricci tensor is the Hessian of a function. In dimensions 2 and 3 for sure, and probably in all higher dimensions too, there exists a complete gradient soliton with bounded curvature and strictly positive curvature operator which is rotationally symmetric around an origin; it can be found by solving an ODE.

Eternal solutions with bounded curvature are important because they occur as models for slowly forming singularities. Our main result is the following start at a classification.

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