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ON SMOOTH RANK-1 MAPPINGS OF BANACH SPACES ONTO THE PLANE

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Abstract

For any separable infinite-dimensional Banach space E we construct a surjective C^∞ mapping $f: E \to \mathbb{R}^2$ satisfying $\operatorname{rank} Df(v) \leq 1$ for all $v \in E$.

A Fréchet differentiable map $f: E \to F$ is called *rank-r* provided $\operatorname{rank} Df(v) \leq r$ for al $v \in E$. Surjective rank-1 mappings $f: \mathbb{R}^n \to \mathbb{R}^m$ are known to exist whenever $n > m > 1$ (see [1], [2], [6], [15]); by the classical Morse-Sard theorem, however, such mappings[1] cannot belong to the smoothness class C^{n-m+1}.

Let E denote a separable infinite-dimensional Banach space. The aim of this note is to construct a C^∞ rank-1 mapping of E onto $\mathbb{R}^2$. Because our technique generalizes easily to produce smooth rank-1 mappings of E onto any higher-dimensional Euclidean space, this settles a recent question of H. Sussmann [14] and Y. Yomdin [15] (see also [4, p. 59]).

To begin our construction, we recall that by a result of Johnson and Rosenthal [5] every separable infinite-dimensional Banach space has a quotient with a Schauder basis.[2] For our purposes, we may therefore assume that E has a bounded basis with corresponding unit coordinate functions $\{\lambda_j\}$ (cf. [11, p. 20f]). The symbol m_k denotes a $k \times k$ matrix with ij-entry $m_k(i, j) \in \{1, 3, 5, 7\}$, and the notation $m_k \prec m_l$ implies $m_k(i, j) = m_l(i, j)$ for $i, j = 1, \cdots, k$.

Cylinder Sets in E

Let $I(a_1, \cdots, a_k)$ denote the set of those $x \in [0, 1]$ such that a_i is the ith digit in the base-9 expansion of x. We define the family $\mathscr{B}$ of

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[1] For a sharper smoothness bound in the context of singular mappings, see [1], [2].

[2] I am indebted to Y. Benyamini for calling the article [5] to my attention. An analogous construction can be carried out using the biorthogonal sequences constructed in [8], [9].