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A FINITENESS THEOREM FOR RICCI CURVATURE IN DIMENSION THREE

SHUN-HUI ZHU

1. Introduction

The purpose of this paper is to prove the following result.

Theorem 1. *There are only finitely many homotopy types in the class of three-dimensional Riemannian manifolds* M *satisfying*

$$\mathrm{Ric}(M) \geq -H^2, \quad \mathrm{Diam}(M) \leq D, \quad \mathrm{Vol}(M) \geq V,$$

where $\mathrm{Ric}(M)$ *is the Ricci curvature,* $\mathrm{Diam}(M)$, *the diameter, and* $\mathrm{Vol}(M)$, *the volume of* M.

As a noncompact counterpart to Theorem 1, we also prove

Theorem 2. *Let* M^3 *be a complete open three-manifold satisfying*

$$\mathrm{Ric} \geq 0, \qquad \mathrm{Vol}(B_p(r)) \geq cr^3.$$

Then M *is contractible.*

Results of the type of Theorem 1, known as finiteness theorems, were first obtained by A. Weinstein [20] and J. Cheeger [4], [12]. Cheeger's finiteness theorem states that there are only finitely many diffeomorphism types for the class of Riemannian manifolds with a bound on the absolute value of sectional curvature and bounds on the diameter and volume identical to that of Theorem 1. Subsequently, Grove-Petersen [7] proved the finite homotopy type theorem only assuming a lower bound on sectional curvature. This was later strengthened to finite diffeomorphism types for $n \neq 3, 4$ by Grove-Petersen-Wu [8]. Theorem 1 is an attempt to generalize this to an assumption on Ricci curvature instead of sectional curvature.

All finiteness theorems as quoted above prove that, under a bound on the curvature (which is local), the topology of the manifold is controlled by its size. The proofs of these theorems rely on the understanding of the local structure of the corresponding class. For Cheeger's finiteness theorem, the crucial step is to prove a lower bound on the injectivity radius for that class, hence on a uniform size smaller than that bound, the topology is simple: it is diffeomorphic to a Euclidean ball. The corresponding statement for the

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