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BRUHAT CELLS IN THE NILPOTENT VARIETY AND THE INTERSECTION RINGS OF SCHUBERT VARIETIES

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1. Introduction

Let G be a complex semisimple Lie group with fixed opposite Borel subgroups B and B^-, and let H be the maximal torus $B \cap B^-$. $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ denote the Lie algebras of G, B, H respectively and $W = N(H)/H$ is the Weyl group of (G, H). A famous result in Lie theory says that the cohomology algebra $H^{\cdot}(G/B; \mathbb{C})$ of the flag variety G/B of G is isomorphic to the coordinate ring $A(\mathcal{N} \cap \mathfrak{h})$ of the scheme-theoretic intersection of the nilpotent variety $\mathcal{N} \subset \mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$. The purpose of this paper is to extend this result to Schubert varieties $X_w := \overline{BwB/B}$ in G/B, where $w \in W$.

We introduce a locally closed stratification $\mathcal{B}_w$ of $\mathcal{N}$ by "Bruhat cells" defined by putting $\mathcal{B}_w = \mathrm{Ad}(Bw^{-1}B)\mathfrak{u}$, where $\mathfrak{u}$ is the nilradical of $\mathfrak{b}$. $\mathcal{N}_w := \overline{\mathcal{B}}_w$ is a Zariski closed irreducible cone in $\mathfrak{g}$ such that $\mathcal{N}_w \subseteq \mathcal{N}_y$ if and only if $X_w \subseteq X_y$. Recall that the scheme-theoretic intersection of varieties Z_1 and Z_2 in $\mathfrak{g}$ is the scheme $Z_1 \cap Z_2$ defined by the ideal $I(Z_1) + I(Z_2)$ where $I(Z_i)$ is the ideal of Z_i in the coordinate $A(\mathfrak{g})$ of $\mathfrak{g}$. By definition, the coordinate ring $A(Z_1 \cap Z_2)$ of $Z_1 \cap Z_2$ is $A(\mathfrak{g})/(I(Z_1) + I(Z_2))$. We will prove

Theorem 1. *For each $w \in W$, there exists a surjective degree doubling homomorphism of graded $\mathbb{C}$-algebras $\psi_w \colon A(\mathcal{N}_w \cap \mathfrak{h}) \to H^{\cdot}(X_w; \mathbb{C})$ such that if $X_w \subseteq X_y$, the diagram*

$$
\begin{array}{ccc}
A(\mathcal{N}_y \cap \mathfrak{h}) & \xrightarrow{\psi_y} & H^{\cdot}(X_y; \mathbb{C}) \\
\downarrow & & \downarrow \\
A(\mathcal{N}_w \cap \mathfrak{h}) & \xrightarrow{\psi_w} & H^{\cdot}(X_w; \mathbb{C})
\end{array}
\tag{1.1}
$$

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